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Design of hyperchaotic flows

G. Baier and S. Sahle
Institute for Chemical Plant Physiology, University of Tübingen, D-72076 Tübingen, Germany
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We present a design strategy in terms of ordinary differential equations which creates chaotic attractors with an increasing number of positive Lyapunov exponents as the (finite) dimension of the system is increased. First, we introduce the most simple abstract equation containing only one nonlinearity. Second, we suggest a piecewise linear version of the abstract equation. Third, we propose a set of chemical reactions and demonstrate that the corresponding rate equations produce hyperchaotic behavior equivalent to the abstract system.

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I. INTRODUCTION

An equation for chaos with two positive Lyapunov characteristic exponents (LCEs) was given in 1979 by Rössler [1]. It contained only one nonlinearity and it was assumed that a hierarchy of higher chaotic attractors might exist in equally simple (with respect to nonlinearity) deterministic equations if the dimension of the equation is increased stepwise. However, the proposed equation did not show generic routes from limit cycle to hyperchaos and it was not possible to extend it to higher dimensions. On the other hand, hyperchaos and the chaotic hierarchy [2] can easily be modeled in discrete maps (i.e., explicit models of cross sections of attractors) if a nonlinear variable in a unimodal map with chaotic behavior is delayed linearly. This is a means to generate maximum hyperchaos in diffeomorphisms with only one nonlinear term [3]. We propose an equally simple mechanism in an abstract ordinary differential equation and demonstrate that a chemical reaction mechanism producing the chaotic hierarchy can be derived.

II. ABSTRACT EQUATION

Consider the N-dimensional system of ordinary differential equations

$$\dot{x}_1 = -x_2 + a\,x_1,$$

$$\dot{x}_i = x_{i-1} - x_{i+1}, \qquad (1)$$

$$\dot{x}_N = \epsilon + b\,x_N\,(x_{N-1} - d),$$

with $x,a,b,d,\epsilon \in \mathbb{R}$; $a>0$; $i=2,\ldots,N-1$; and $N \in \mathbb{N}$.

The linear subsystem $(x_1,\ldots,x_{N-1})$ is a chain of harmonic oscillators for $a=0$. The fixed point condition leads to $x_1=x_i=0$, when N is odd; and to $x_1=x_3=\cdots=x_{N-1}$ and $x_2=x_4=\cdots=x_{N-2}=0$, when N is even. If $a>0$, the antidissipative term $a\,x_1$ leads to an expanding spiraling flow in $(N-1)$ directions with $(N-1)/2$ pairs of complex conjugated eigenvalues for odd N. For even N, $a>0$ leads to a spiraling expansion in $(N-2)$ directions with $(N-2)/2$ pairs of complex eigenvalues with positive real part plus one expanding direction with (positive) real eigenvalue. Variable x_N contains a constant ϵ, one nonlinear expression of second order, and linear dissipation. In accordance with Rössler's equation x_N acts as a threshold variable. Depending on the value of x_{N-1} it either shrinks or grows. Finally, variable x_N is linearly coupled to the equation for x_{N-1} to yield the complete system. Equation (1) thus contains only one nonlinear term.

For $N=3$, Eq. (1) is Rössler's equation for chaotic flows [4]. For $N=4$, Eq. (1) is a prototypical system for hyperchaotic flows with two positive LCEs. The linear three-variable subsystem (x_1,x_2,x_3) possesses eigenvalues

$$\lambda_1=0,\ \lambda_{2,3}=\pm i\,\sqrt{2},$$

for $a=0$. If $a>0$, the subsystem possesses one positive real eigenvalue and one pair of complex eigenvalues with positive real part. The flow expands along the axis $x_1=x_3$ in a spiral-like manner. The growth in variable x_3 activates the nonlinear switch x_4. For certain sets of parameters x_N keeps the flow bounded and leads to attractive solutions. As in the system with $N=3$, the nonlinearity may create chaotic mixing. However, in the four-dimensional system two directions of stretching and folding are possible. For example, for the set of parameters $b=4, d=2, \epsilon=0.1$ a sequence of bifurcations from fixed point to hyperchaos is observed as parameter a is increased starting at $a=0$. First, a limit cycle is created in a Hopf bifurcation and the limit cycle loses stability in a secondary Hopf bifurcation to give rise to an attracting two-torus. The quasiperiodic motion is followed by a locked mode on the two-torus. The locked mode loses stability in a secondary Hopf bifurcation and wrinkling of the two-torus leads to chaos on a fractal torus. In the region of chaotic behavior we find a small window of chaos with one positive LCE and a broader window with hyperchaos. For $a=0.4$ we calculated the following spectrum of LCEs (numerical values in bits per time unit): (0.146, 0.071, 0, −1.80), and thus two positive exponents. The cross section is sheetlike and shows several foldings.

For $N=5$, Eq. (1) possesses attractors with three positive LCEs. The linear subsystem $(x_1,\ldots,x_4)$ possesses eigenvalues

$$\lambda_{1,2,3,4}=\pm i\,(3/2\pm\sqrt{5/4})^{1/2},$$

for $a=0$. If $a>0$, the subsystem possesses two pairs of complex eigenvalues with positive real part. The flow takes place on an expanding two-torus. Again, the nonlinear variable

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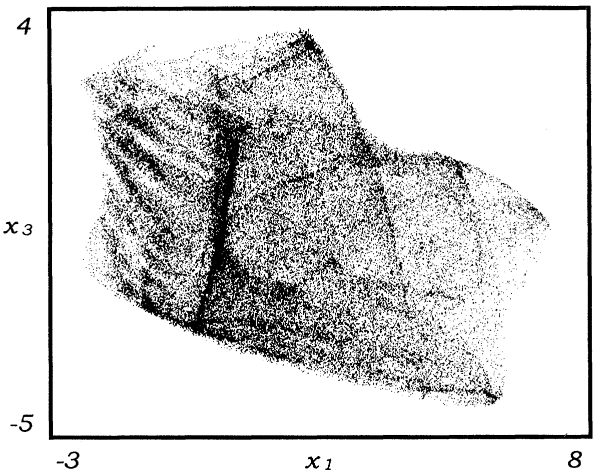

FIG. 1. Second-order cross section of higher chaotic flow in Eq. (1) with $N=5$ and $a=0.33$, $b=4$, $d=2$, $\epsilon=0.1$. The first-order cross section was taken at $x_5=1$. For the second cut the slice $x_2=0\pm0.07$ of the cross section was plotted.

may lead to bounded solutions in the complete system and creates positive LCEs by means of the stretching and folding procedure. Starting with a stable fixed point solution at $a=0$, two successive Hopf bifurcations lead to a limit cycle and an attracting two-torus, respectively. As a is increased further, the bifurcation diagram shows lockings, chaos, and hyperchaos. There are crises, and several regions of multistability with complicated basin boundary structure. Qualitatively, the bifurcation diagram of Poincaré cross sections of the flow resembles the corresponding diagram of the four-dimensional (4D) diffeomorphism in [5] which also possesses only one nonlinearity. The spectrum of LCEs for the higher chaotic attractor at $a=0.33$ (other parameters: $b=4, d=2, \epsilon=0.1$) was numerically calculated as (0.141, 0.108, 0.048, 0, -9.95) and thus three positive LCEs, i.e., maximum hyperchaos for $N=5$. The Poincaré cross section of this attractor has a Lyapunov dimension $3<D_\lambda<4$ and appears as a fuzzy cloud of points. Figure1 is a second-order cross section which shows a folded sheetlike structure predicted from the study of 4D diffeomorphisms [3].

We calculated the spectrum of LCEs for Eq. (1) with $N=7$ at $a=0.32$, $b=4$, $d=2, \epsilon=0.1$ and found (0.102, 0.079, 0.067, 0.035, 0, -0.024, -9.95). There are four positive LCEs, one less than the maximal possible number. For $N=9$, we calculated six positive LCEs (0.078, 0.066, 0.057, 0.043, 0.027, 0.010, 0, -0.024, -9.63) at $a=0.30$, $b=4$, $d=2$, $\epsilon=0.1$. Thus the number of positive LCEs keeps increasing as N is increased. If all parameters are kept constant the numerical value of the first exponent decreases as N is increased and the sum of positive LCEs (as an estimation for the metric entropy) stays nearly constant as N is increased (cf. [6] for a related observation).

If the equation for x_N in system (1) is substituted by

$$\dot{x}_N=b\,(|x_{N-1}-d|+x_{N-1}-d)-x_N \tag{2}$$

then a piecewise linear version is obtained. For $N=5, a=0.30, b=4, d=2$ the following spectrum of LCEs was calculated: (0.123, 0.096, 0.062, 0, -1.16). Again, three positive LCEs demonstrate maximum hyperchaos.

III. CHEMICAL EQUATION

One major field where low-dimensional chaotic events have been observed experimentally is chemical kinetics. Isothermal, well-stirred reactions with continuous supply of substrate can be modeled successfully by a finite set of deterministic differential equations. Consider the following set of reactions:

$$X_1 \rightarrow 2\,X_1$$

$$2\,X_1 \xrightarrow{a} 3\,X_1$$

$$X_{i-1}+X_i \rightarrow 2\,X_i$$

$$X_{N-1} \rightarrow$$

$$X_{N-1}+X_N \xrightarrow{b} 2\,X_N$$

$$\xrightarrow{\epsilon} X_N$$

$$X_N \xrightarrow{c}$$

with $i=2,\ldots,N-1$, $N\in\mathbb{N}$. X_j denotes chemical species. Species X_N is supplied at a constant rate, and species X_N and X_{N-1} are assumed to decay in a first-order reaction. The nonlinearities stem from the autocatalytic reactions which are assumed to be second-order reactions. The initial conditions of species X_1 through X_N have to be nonzero.

Applying the laws of mass-action the dynamics of this scheme is described by the following set of ordinary differential equations:

$$\begin{aligned} \dot{x}_1&=x_1-x_1x_2+a\,x_1^2,\\ \dot{x}_i&=x_{i-1}x_i-x_ix_{i+1},\\ \dot{x}_{N-1}&=x_{N-2}x_{N-1}-x_{N-1}-b\,x_Nx_{N-1},\\ \dot{x}_N&=\epsilon+b\,x_N(x_{N-1}-d), \end{aligned} \tag{3}$$

with $x,a,b,d,\epsilon\in\mathbb{R}$, $c=bd$; $i=2,\ldots,N-1$; and $N\in\mathbb{N}$. Here, x_j denotes the concentration of species X_j.

Equation (3) is a chemical system which works analogously to the abstract system Eq. (1). There is a phase-space-volume preserving set of reactions in the subsystem $(x_1,\ldots,x_{N-1})$ for $a=0$. This subsystem is a generalized Lotka-Volterra scheme for a sequence of autocatalytic reactions with conservative periodic or quasiperiodic dynamics. For $a>0$, the second-order autocatalytic term in the equation for x_1 leads to antidissipative spiral-like blowup of the flow. Switching variable x_N can keep the solutions bounded for certain ranges of parameters and introduces dissipation to the system. x_N is coupled to the Lotka-Volterra subsystem by means of an additional autocatalytic reaction. Equation (3) possesses chaotic attractors for $N=3$ and attractors with two positive LCEs for $N=4$. As parameter a is increased (starting from $a=0$), for $N=3, b=10, d=2.4, \epsilon=0.01$ there is a Hopf bifurcation from fixed point to limit cycle followed by a period-doubling sequence of the limit cycle to chaos. As in the Rössler equation, spiral chaos and Shilnikov chaos can be distinguished. If parameter a is increased (starting at $a=0$) in Eq. (3) with $N=4, b=10, d=2.4, \epsilon=0.01$, the following sequence of attractors is observed: fixed point—limit cycle—quasiperiodic motion on a two-torus—locked mode on a two-torus—chaos on a fractalized two-torus—hyperchaos. Qualitatively, the chemical equation (3) thus reproduces the bifurcation sequence of the abstract equation (1). Also, this sequence and the hyperchaotic attractor are found in a broad region of parameters b, d, and ϵ. With $N=5$ and the set of parameters $a=0.3$, $b=10$, $d=2.4$, $\epsilon=0.01$, the spectrum of LCEs is (0.135, 0.089, 0.030, 0, -14.87), and thus three positive LCEs. Again, the bifurcation diagram was found to be qualitatively the same as in the abstract system Eq. (1) if parameter a is varied at fixed values of b, d, and ϵ. The attractor with three positive LCEs exists in a broad region of parameters.

The chemical model suggests that the design principle of Eq. (1) can be implemented in homogeneous chemical or biochemical reactions. Autocatalytic sets of reactions with nonlinear feedback show features of self-organized metabolism [7] and they may serve the study of the role of chaotic dynamics in information processing.

Hyperchaos with two positive LCEs was observed experimentally in a periodically driven NMR laser [8], avalanche breakdown of p-germanium [9], and during the catalytic CO oxidation on a platinum single crystal [10]. Killory *et al.* found two positive LCEs in a 4D chemical system derived from Rössler's equation [11]. Chaos with three positive LCEs in continuous systems was observed numerically in a model of bacteria-phage interaction [12], in a chemical reaction chain with nonlinear feedback control [13], and in diffusively coupled biochemical oscillators [14]. In the present contribution we presented the fundamental design of higher chaos and the chaotic hierarchy in a prototypical ordinary differential equation, in a piecewise linear version for future analytic studies, and in a system composed of chemical reactions as a model application.

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